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SENSITIVITY OF THERMOMECHANICAL RESPONSE TO THERMAL BOUNDARY CONDITIONS AND MATERIAL CONSTANTS

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Abstract—Forced vibrations of a thermoviscoelastic material give rise to nonlinear coupling between thermal and mechanical responses. In particular, the response of a rod of solid rocket propellant to axial vibrations includes several narrow frequency ranges characterized by rapid or even discontinuous increases in thermomechanical response. In this paper, the sensitivity of these critical frequency ranges to boundary condition modeling and to small changes in constitutive properties and ambient conditions is studied. It is found that the solution is somewhat sensitive to the temperature dependence in the thermoviscoelastic material law, but far more sensitive to the type of thermal boundary conditions imposed upon the problem.

INTRODUCTION

The thermomechanically coupled response of a viscoelastic rod subjected to an axially imposed vibration was studied by Huang and Lee[1]. Mukherjee[2] investigated the post-transience response and showed that there exist several critical driving frequencies. Stress and temperature levels in response to driving frequencies just higher than critical were found to be much higher than those calculated for a slightly lower, subcritical frequency. Mukherjee's analysis was based upon a stress boundary condition at the driven end of the rod. It was later shown [3] that specification of a displacement boundary condition at the driven end does not lead to a one-to-one correspondence with the specified-stress problem, although the same "critical-frequency" phenomenon is observed.

This paper addresses itself to the sensitivity of the calculated response to various parameters in the problem. The most important manifestation of a change in the thermomechanical response of the rod is the shifting of the critical frequency. The sensitivity of the critical frequencies is studied with respect to:

1. The thermal boundary conditions at the driven end.
2. Small changes in the ambient temperature.
3. Small changes in various parameters in the constitutive equation for the particular solid propellant being studied.

STATEMENT OF THE PROBLEM

The coupled, second-order, non-linear ordinary differential equations governing the one-dimensional thermomechanical response of a rod of thermoviscoelastic material are:

$$\mathbf{U} - \left(\frac{\gamma\tau'}{\tau}\right)\mathbf{U}' + a_4\tau^{\gamma}\begin{bmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{bmatrix}\mathbf{U} = 0 \tag{1}$$

$$\tau'' + \left(\frac{a_5}{c_1 a_4}\right)\tau^{-\gamma}[(U_1')^2 + (U_2')^2] = 0 \tag{2}$$

where

$\mathbf{U} = \dfrac{1}{L}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

L = length of the rod

u_1, u_2 = components of displacement in- and out-of-phase with the driving displacement

$\tau = (T - T_1)/(T_0 - T_1)$

T = absolute temperature of the material

T_0 = ambient temperature
$'$ = d/dq, q = normalized position coordinate along the rod ($q = 0$ at free end, $q = 1$ at driven end)
$a_4 = \rho L^2 \omega^{2+\beta}(T_0 - T_1)^\gamma$
$a_5 = c_1 c_2 \rho \omega^3 L^4 / [2c_1{}^2 + c_2{}^2)\kappa(T_0 - T_1)]$
ρ = density of the rod
κ = thermal conductivity of the rod
ω = the driving frequency

and the complex Young's modulus of the rod is given by

$$E^* = E_1 + iE_2$$

where

$$E_1 = c_1(c_1{}^2 + c_2{}^2)^{-1}\omega^{-\beta}(T - T_1)^{-\gamma} \qquad T > T_1 \tag{3a}$$

$$E_2 = (c_2/c_1)E_1 \tag{3b}$$

and

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4.61 \\ 1.62 \end{pmatrix} \times 10^{-11}\,(\text{psi})^{-1}\,(\text{sec})^{-\beta}(^\circ\text{F})^\gamma$$

$$\beta = -0.214$$

$$\gamma = 3.21$$

$$T_1 = -125^\circ\text{F}.$$

Equations (1) and (2) are derived in Young[3]. They are solved by iteration and finite differences.

BOUNDARY CONDITIONS

The boundary conditions applied to this problem in[3] are:

$$\mathrm{U}'(0) = 0 \tag{4}$$

$$\tau(0) = 1 + \frac{\kappa}{HL}\tau'(0) \tag{5}$$

where H = surface conductance, and

$$\mathrm{U}(1) = \frac{1}{L}\begin{Bmatrix} u_d \\ 0 \end{Bmatrix} \tag{6}$$

where u_d = amplitude of driving displacement, and

$$\tau(1) = 1. \tag{7}$$

Equation (7) is predicated upon the assumption that the mechanism which forces the vibration of the rod acts as a perfect heat sink and that it remains at the ambient temperature. If, however, it is allowed to warm up along with the rod, a different boundary condition becomes appropriate.

The thermal conductivity of most metals or ceramics is much higher than that for the rod itself. Thus, even for a severe temperature gradient within the rod at the driven end, thermal equilibrium can be maintained with a very slight thermal gradient within the driving mechanism. Treating the temperature within the mechanism as being uniform and equal to that at the driven end of the rod, a surface conductance boundary condition can be applied. Therefore, eqn (7) is replaced with

$$\tau(1) = 1 - \frac{\kappa}{HL}\tau'(1). \tag{8}$$

The term $\kappa\tau'(1)$ comes from the heat applied to the driver and is evaluated for the rod at its driven end. Neglecting the small drop in temperature across the driver, Newton's law of cooling is applied using the end temperature of the rod itself. A value of $\kappa/HL = 1$, as given in [2], is used in both thermal boundary conditions, eqns (5) and (8).

The effect of this modification is to reduce the heat flow through the driven end and to significantly raise the temperature profile along the rod. A comparison of results using the two boundary conditions is given in Fig. 1.

In addition to raising the overall steady-state temperature, the modified boundary condition also leads to a marked lowering of the critical frequencies. This phenomenon and its implications are shown in Fig. 2, where the maximum temperature maintained within the rod is

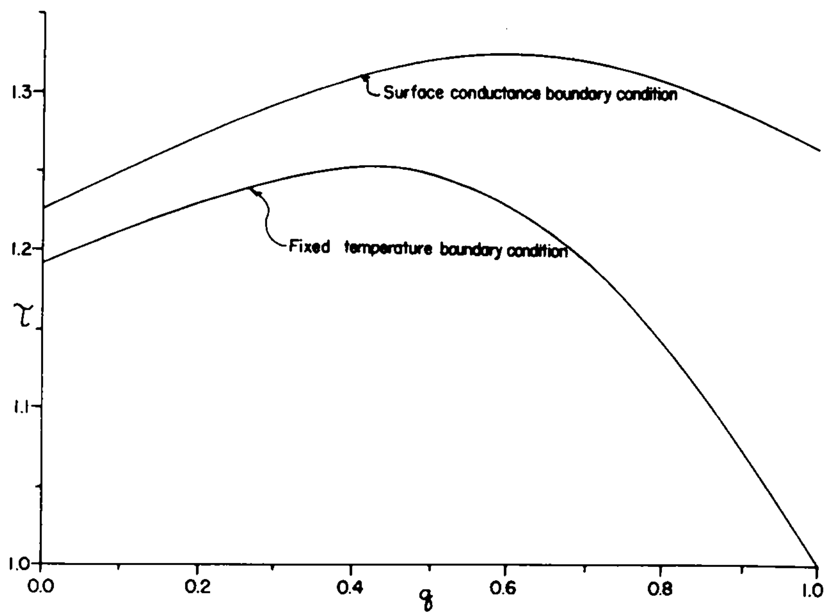

Fig. 1. Dimensionless temperature profiles for $\omega = 4000$ rad/sec.

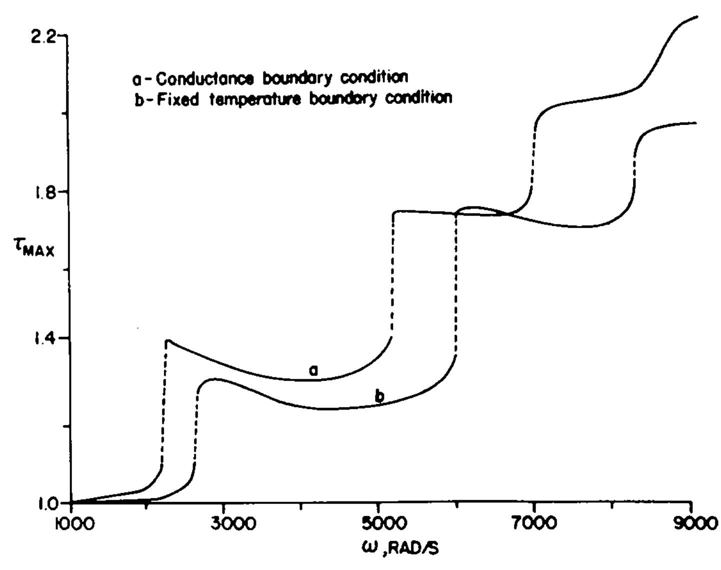

Fig. 2. τ_{max} vs ω for two thermal boundary conditions.

plotted as a function of driving frequency. The three critical frequencies found for the fixed-temperature boundary condition are reduced by approximately 400 rad/sec, 800 rad/sec and 1300 rad/sec respectively. Additionally, a fourth region of rapidly increasing response is predicted at frequencies close to the previous third discontinuity. The most dramatic effect upon the predicted response occurs for those frequencies which were passed over as the critical frequencies shifted to lower values. The increased temperature associated with the greater impedance to heat flow (as shown in Fig. 1 for $\omega = 4000$ rad/sec) is quite small compared to the increase associated with being shifted to the other side of a critical frequency. Compare $\omega = 4000$ and $\omega = 5500$ in Fig. 2. Furthermore, these shifts are sufficiently large to affect nearly 30% of the frequency band studied.

EFFECT OF AMBIENT TEMPERATURE

The pronounced non-linearities in the problem have been seen to result in a non-correspondence between stable responses to known-stress and known-displacement boundary conditions [2, 3] and to exacerbate the differences resulting from various postulated thermal boundary conditions at the driven end.

This logically calls into question the sensitivities of the response to other parameters, such as the ambient temperature. To test for the presence of any large shifts in the critical frequencies, the eqns (1) and (2) were solved for $T_0 = 60°F$ and $70°F$ and the results were compared with the initial data for 65°F. The comparison is shown in Fig. 3. The dimensionless variable τ is based on $T_0 = 65°$ for all three cases. The response temperature is seen to have been shifted up or down directly with T_0 only for $\omega < 2500$ rad/sec. At this point, the nonlinearities of the problem take over and the simple pattern is lost. In fact, over a wide range ($3500 < \omega < 5500$ rad/sec) the response for $T_0 = 60°$ is warmer than that for either 65° or 70°. At higher frequencies the three responses are virtually indistinguishable. Most importantly, there is very little shifting of the critical frequencies. However, some slight shifting is apparent ($\delta\omega \leq 100$ rad/sec) and, given the highly nonlinear nature of the responses seen so far, it would be inappropriate to suggest that shifting could not be more pronounced in a different temperature range.

SENSITIVITY TO MATERIAL PARAMETERS

As a final check on the sensitivity of the response of the rod to slight variations in the modeling of the problem, the roles of two of the important material parameters, β and γ, in eqn (3a) were tested. Solutions were generated for deviations of approximately 1% in the exponents β and γ.

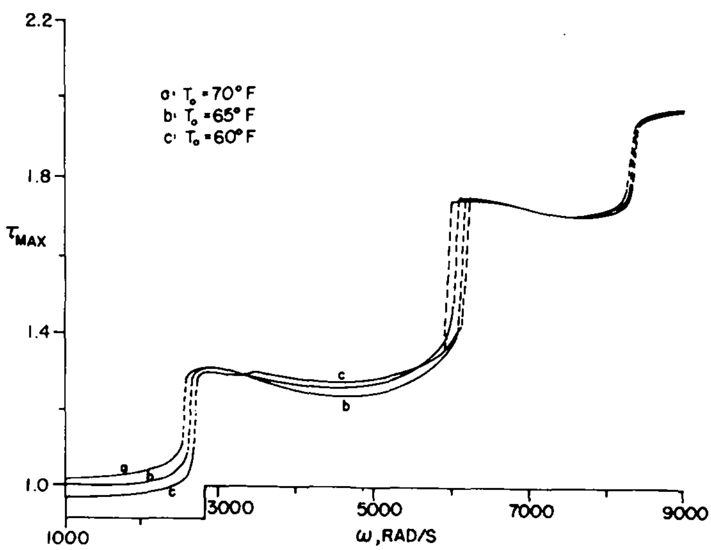

Fig. 3. τ_{max} vs ω for three ambient temperatures.

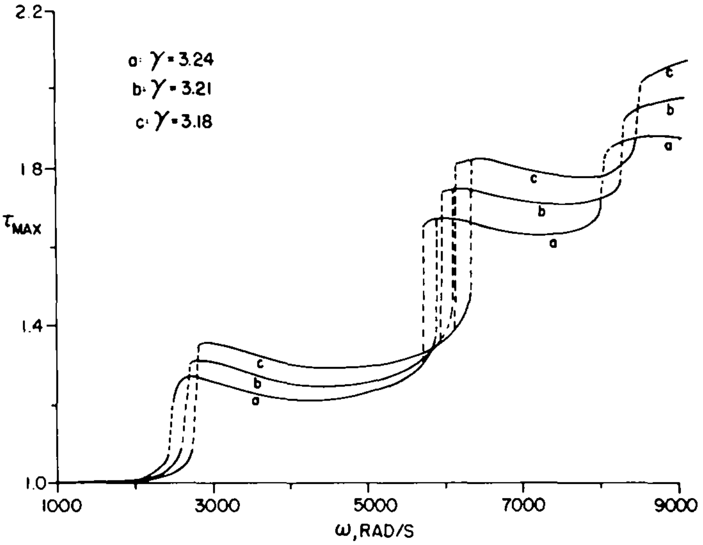

Fig. 4. τ_{max} vs ω for three values of the temperature exponent.

It is found that γ, the exponent of temperature, has a much more pronounced effect than does the ambient temperature. A 1% increase in γ, from 3.21 to 3.24, lowers the critical frequencies by 200–300 rad/sec and lowers the overall response temperature between critical frequencies to a much greater degree than does a 5°F shift in ambient temperature. The comparison of solutions for $\gamma = 3.18$, 3.21 and 3.24 is shown in Fig. 4.

A similar analysis was performed with variations in β, the frequency exponent in the material law. Solutions for $\beta = -0.212$, -0.214 and -0.216 were calculated and the results, when plotted to the same scale used in Figs. 2–4, show almost undetectable changes. Critical frequencies change by less than 40 rad/sec and dimensionless maximum temperatures by less than 0.02. The trend as β increases in absolute value is to slightly raise the critical frequencies as well as the overall temperature.

CONCLUSIONS

While some sensitivity of the solutions to material constants, especially the temperature exponent, is found, it is seen that the overriding concern must be the appropriateness of the boundary conditions, both mechanical and thermal. Mukherjee[2], Young[3] and this paper provide comparisons between two different mechanical boundary conditions and two different thermal boundary conditions and serve to illustrate the importance of properly treating this early step in the solution of nonlinear thermomechanical coupling problems.

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